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**Abstract**—In this note, a general existence theorem of generalized variational inequalities for quasi-monotone set-valued mappings in locally convex topological vector spaces has been established. Our result includes corresponding results in recent literature as special cases.

**Keywords**—Set-valued quasi-monotone, Ky Fan minimax principle, Generalized variational inequality, 0-diagonal concave.

## 1. INTRODUCTION

Throughout this paper, for simplicity, all topological vector spaces (respectively, locally convex topological vector spaces) are assumed to be *Hausdorff*. Let  $X$  be a nonempty set; we shall denote by  $2^X$  and  $\mathcal{F}(X)$  the family of all subsets of  $X$  and the family of all nonempty finite subsets of  $X$ . If  $E$  is a topological vector space, we shall denote by  $E^*$  the topological dual of  $E$  and by  $\operatorname{Re}\langle x, u \rangle$  the real part of the pairing between  $x \in E$  and  $u \in E^*$ . Let  $K$  be a nonempty convex subset of  $E$  and  $f : K \rightarrow 2^{E^*}$  a set-valued mapping with nonempty values. The *generalized variational inequality problem*  $\operatorname{GVIP}(f, K)$  is to find  $\hat{y} \in K$  and  $\hat{v} \in f(\hat{y})$  such that

$$\operatorname{Re}\langle x - \hat{y}, \hat{v} \rangle \geq 0, \quad \text{for all } x \in K. \quad (1)$$

In order to study the existence of solutions of *generalized complementarity problems* (GCP( $f, K$ )) for quasi-monotone mappings (see definition below) in general settings, Ding [1, Theorem 2.2] recently proved the following existence theorem of solutions for generalized variational inequalities (the proof in an English version can also be found in the Appendix of [2]).

**THEOREM A.** Let  $K$  be a nonempty convex subset of a Hausdorff locally convex topological vector space  $E$  which is of second category. Let  $f : K \rightarrow 2^{E^*}$  be a quasi-monotone mapping such that  $f$  is upper semicontinuous from the line segments in  $K$  to the weak\* topology of  $E^*$  and each  $f(x)$  is nonempty compact in the strong topology on  $E^*$ . Suppose that there exist nonempty weakly compact convex subset  $K_0$  of  $K$  and nonempty weakly compact subset  $D$  of  $K$  such that for each  $y \in K \setminus D$ , there exists  $x \in \text{co}(K_0 \cup \{y\})$  with

$$\operatorname{Re}\langle y - x, u \rangle > 0, \quad \text{for all } u \in f(x).$$

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Then there exists a point  $\hat{y} \in K$  such that

$$\inf_{v \in f(\hat{y})} \operatorname{Re}\langle \hat{y} - x, v \rangle \leq 0, \quad \text{for all } x \in K.$$

If, in addition,  $f(\hat{y})$  is also convex, then there exists  $\hat{v} \in f(\hat{y})$  such that

$$\operatorname{Re}\langle x - \hat{y}, \hat{v} \rangle \geq 0, \quad \text{for all } x \in K,$$

that is,  $(\hat{y}, \hat{v})$  is a solution of  $\text{GVIP}(f, K)$ .

We recall that a topological space  $X$  is said to be of *second category* if  $X$  cannot be expressed as the union of a sequence of nowhere dense sets (see [3, p. 41]). Of course, each Banach space is of second category. However, a topological vector space (respectively, a locally convex topological vector space) need not be of second category in general.

As Theorem A above has found many applications in the study of variational theory itself, mathematical programming and operations research such as complementarity problems, and so on, it is our purpose in this note to generalize Theorem A to locally convex topological vector spaces (hence, which need not be of second category). Our generalization of  $\text{GVIP}(f, K)$  includes corresponding results of Cottle and Yao [4], Saigal [5], Fang and Peterson [6], Harker and Pang [7], Siddiqi and Ansari [8], Shih and Tan [9], Tan and Yuan [10], and Zhou and Chen [11] as special cases.

Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow 2^Y$ . Then  $f$  is said to be upper semicontinuous (in short, u.s.c.) on  $X$  if for each  $x_0 \in X$  and any open set  $V$  in  $Y$  containing  $f(x_0)$ , there exists an open neighborhood  $U$  of  $x_0$  in  $X$  such that  $f(x) \subset V$  for all  $x \in U$ . If  $X$  is a topological space and  $f : X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  is an extended real valued function. Then  $f$  is said to be *compactly lower semicontinuous* if  $f$  is lower semicontinuous on each nonempty compact subset of its domain  $X$ .

Let  $K$  be a nonempty subset of a topological vector space  $E$  and  $f : K \rightarrow 2^{E^*}$  a set-valued mapping with nonempty values. Then  $f$  is said to be

- (1) *monotone* if for each  $x, y \in K$ ,  $u \in f(x)$ , and  $v \in f(y)$ ,  $\operatorname{Re}\langle x - y, u - v \rangle \geq 0$ ;
- (2) *pseudo-monotone* (see [5, p. 263, Definition 3.4]) if for each  $x, y \in K$ ,  $u \in f(x)$ , and  $v \in f(y)$ ,  $\operatorname{Re}\langle x - y, v \rangle \geq 0$  implies  $\operatorname{Re}\langle x - y, u \rangle \geq 0$ ;
- (3) *quasi-monotone* (compare with [1]) if for each  $x, y \in K$ ,

$$\inf_{v \in f(y)} \operatorname{Re}\langle x - y, v \rangle > 0 \quad \text{implies} \quad \inf_{u \in f(x)} \operatorname{Re}\langle x - y, u \rangle > 0.$$

Clearly, each monotone mapping is pseudo-monotone, and Proposition 3.2 of Karamardian and Schaible [12, p. 39] shows that the definition (3) above is a set-valued generalization of the definition (2) for a single-valued mapping.

Let  $X$  be a nonempty convex subset of a topological vector space and  $\psi : X \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  an extended real-valued function. We also recall that (see [11])  $\psi$  is said to be *0-diagonal quasi-concave in  $y$*  if for each  $A \in \mathcal{F}(X)$  and  $x_0 \in \operatorname{co}(A)$ , we have  $\min_{y \in A} \psi(x_0, y) \leq 0$ .

Let  $\psi(x, x) \leq 0$  for each  $x \in X$ . It is clear that if  $\psi(x, y)$  is quasi-concave in  $y$  for each  $x \in X$ ,  $\psi(x, y)$  is 0-diagonal concave in  $y$ ; but the converse does not hold by Remark 2.2 of Zhou and Chen [11, p. 215].

## 2. GENERALIZED VARIATIONAL INEQUALITIES

The aim of this section is to establish the following existence theorem of  $\text{GVIP}(f, K)$  for quasi-monotone set-valued mappings in locally convex Hausdorff topological vector spaces.

**THEOREM 2.1.** *Let  $K$  be nonempty convex subset of a locally convex Hausdorff topological vector space  $E$  with  $E^* \neq \emptyset$  (hence,  $E$  need not be of second category). Let  $f : K \rightarrow 2^{E^*}$  be*

a quasi-monotone mapping such that  $f$  is upper semicontinuous from the line segments in  $K$  to the weak\* topology of  $E^*$  and each  $f(x)$  is nonempty compact in the strong topology on  $E^*$ . Suppose that there exist nonempty weakly compact convex subset  $K_0$  of  $K$  and nonempty weakly compact subset  $D$  of  $K$  such that for each  $y \in K \setminus D$ , there exists  $x \in \text{co}(K_0 \cup \{y\})$  with

$$\text{Re}\langle y - x, u \rangle > 0, \quad \text{for all } u \in f(x).$$

Then there exists a point  $\hat{y} \in K$  such that

$$\inf_{v \in f(\hat{y})} \text{Re}\langle \hat{y} - x, v \rangle \leq 0, \quad \text{for all } x \in K.$$

If, in addition,  $f(\hat{y})$  is also convex, then there exists  $\hat{v} \in f(\hat{y})$  such that

$$\text{Re}\langle x - \hat{y}, \hat{v} \rangle \geq 0, \quad \text{for all } x \in K,$$

that is,  $(\hat{y}, \hat{v})$  is a solution of  $\text{GVIP}(f, K)$ .

REMARK 2.1. Theorem 2.1 shows that Theorem A above still holds without the assumption that  $E$  is of secondary category, thus Theorem 2.1 includes Theorem A of [1] as a special case.

In order to prove Theorem 2.1, we first need the following generalization of the Ky Fan minimax principle which is due to Ding and Tan [13].

LEMMA 2.1. Let  $K$  be a nonempty convex subset of a Hausdorff topological vector space  $E$  and  $\psi : X \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be an extended real-valued function such that

- (1) for each fixed  $x \in K$ ,  $y \mapsto \psi(x, y)$  is compactly lower semicontinuous;
- (2) for each  $A \in \mathcal{F}(X)$  and  $y_0 \in \text{co}(A)$ ,  $\min_{x \in A} \psi(x, y_0) \leq 0$ ;
- (3) there exist a nonempty compact and convex subset  $K_0$  of  $K$  and a nonempty compact subset  $D$  of  $K$  such that for each  $y \in K \setminus D$ , there exists  $x \in \text{co}(K_0 \cup \{x\})$  with  $\psi(x, y) > 0$ .

Then there exists  $\hat{y} \in D$  such that  $\sup_{x \in X} \psi(x, \hat{y}) \leq 0$ .

PROOF. It is Theorem 1 of Ding and Tan [13, p. 235]. ■

We also need the following result which was originally proved by Tan and Yuan [10, Lemma 8]; and for completeness, we include its proof here.

LEMMA 2.2. Let  $K$  be a nonempty convex subset of a topological vector space  $E$  and  $T : K \rightarrow 2^{E^*}$  be upper semicontinuous from the line segments in  $K$  to the weak\* topology of  $E^*$  such that each  $T(x)$  is nonempty weak\*-compact. If  $\hat{y} \in K$ , then the inequality

$$\inf_{u \in T(x)} \text{Re}\langle u, \hat{y} - x \rangle \leq 0, \quad \text{for all } x \in K$$

implies the inequality

$$\inf_{w \in T(\hat{y})} \text{Re}\langle w, \hat{y} - x \rangle \leq 0, \quad \text{for all } x \in K.$$

PROOF. Let  $x \in K$  be arbitrarily fixed and  $z_t = tx + (1 - t)\hat{y} = \hat{y} - t(\hat{y} - x)$  for each  $t \in [0, 1]$ . Since  $K$  is convex,  $z_t \in K$  for all  $t \in [0, 1]$ . Thus for all  $t \in (0, 1]$ ,

$$t \cdot \inf_{u \in T(z_t)} \text{Re}\langle u, \hat{y} - x \rangle = \inf_{u \in T(z_t)} \text{Re}\langle u, \hat{y} - z_t \rangle \leq 0$$

so that

$$\inf_{u \in T(z_t)} \text{Re}\langle u, \hat{y} - x \rangle \leq 0, \tag{2}$$

for all  $t \in (0, 1]$ . If  $\inf_{w \in T(\hat{y})} \text{Re}\langle w, \hat{y} - x \rangle > 0$ , let  $G := \{w \in E^* : \text{Re}\langle w, \hat{y} - x \rangle > 0\}$ . Then  $G$  is a weak\*-open set in  $E^*$  such that  $T(\hat{y}) \subset G$ . As  $z_t \rightarrow \hat{y}$  when  $t \rightarrow 0^+$ , by the upper semicontinuity

of  $T$  on  $\{z_t : t \in [0, 1]\}$ , there exists  $t_0 \in (0, 1]$  such that  $T(x_t) \subset G$  for all  $t \in (0, t_0)$ . Note that  $T(z_t)$  is weak\* compact,  $\inf_{u \in T(z_t)} \operatorname{Re}\langle u, \hat{y} - x \rangle > 0$  for all  $t \in (0, t_0)$  which contradicts (2). Thus we must have  $\inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle \leq 0$  and the proof is complete.  $\blacksquare$

LEMMA 2.3. *Let  $K$  be a nonempty convex subset of a topological vector space  $E$  and  $f : K \rightarrow 2^{E^*}$  a set-valued quasi-monotone mapping with nonempty values. Then the mapping  $\psi : K \times K \rightarrow \mathbb{R} \cup \{\infty, +\infty\}$  defined by*

$$\psi(x, y) := \inf_{u \in f(y)} \operatorname{Re}\langle u, x - y \rangle,$$

for each  $(x, y) \in K \times K$  is 0-diagonal concave in  $y$  for each fixed  $x \in K$ .

PROOF. The conclusion follows straightforward from the definitions of quasi-monotone and 0-diagonal concave mappings.  $\blacksquare$

By combining Lemmas 2.1, 2.2, and 2.3, we can now prove Theorem 2.1.

PROOF OF THEOREM 2.1. Define a mapping  $\varphi : K \times K \rightarrow \mathbb{R}$  by

$$\varphi(x, y) = \inf_{u \in f(x)} \operatorname{Re}\langle u, y - x \rangle,$$

for each  $(x, y) \in K \times K$ . Then we have

- (a) for each fixed  $y \in K$ ,  $\varphi(x, y)$  is 0-diagonal concave in  $x$  by Lemma 2.3 as  $f$  is quasi-monotone;
- (b) for each  $x \in K$ , the function  $y \mapsto \varphi(x, y)$  is weakly lower semicontinuous on each weakly compact subset  $A$  of  $K$ .

Note that  $E$  is locally convex, and for each  $x \in K$  and for any nonempty weakly compact subset  $A$  of  $K$ ,  $A - x$  is weakly bounded and hence, by [3, Theorem 3.18],  $A - x$  is strongly bounded in  $E$ . Now we only need to prove that for each fixed  $x \in K$  and for any real number  $\lambda \in \mathbb{R}$  the set  $A_\lambda = \{y \in A : \inf_{u \in f(x)} \operatorname{Re}\langle y - x, u \rangle \leq \lambda\}$  is weakly closed. Let  $\{y_\alpha\}_{\alpha \in \Gamma} \subset A_\lambda$  be a net,  $y_0 \in A$  and  $y_\alpha \rightarrow y_0$ . Since  $f(x)$  is strong-compact, for each  $\alpha \in \Gamma$ , there exists an  $u_\alpha \in f(x)$  such that

$$\operatorname{Re}\langle y_\alpha - x, u_\alpha \rangle = \inf_{u \in f(x)} \operatorname{Re}\langle y_\alpha - x, u \rangle \leq \lambda,$$

and there exists a subnet  $\{u_\beta\}$  of  $\{u_\alpha\}_{\alpha \in \Gamma}$  such that  $u_\beta \rightarrow u_0 \in f(x)$ . Let  $B = \{y_\alpha - y : \alpha \in \Gamma\} \cup \{y_0\}$ . Then  $B$  is strongly bounded. For any given  $\varepsilon > 0$ , let  $W(B, \varepsilon) = \{u \in E^* : \sup_{y \in B} |\langle y, u - u_0 \rangle| < \varepsilon/3\}$ , then  $W = W(B, \varepsilon)$  is a strongly open neighborhood of  $u_0$  in  $E^*$ . Let  $U = \{y \in A : |\langle y - y_0, u \rangle| < \varepsilon/3\}$ . It follows that  $U$  is a weakly open neighborhood of  $y_0$  in  $A$ . Choose  $\beta_0 \in \Gamma$  such that for all  $\beta \geq \beta_0$ ,  $u_\beta \in W$  and  $y_\beta \in U$ . It follows that

$$\begin{aligned} |\operatorname{Re}\langle y_\beta - x, u_\beta \rangle - \operatorname{Re}\langle y_0 - x, u_0 \rangle| &\leq |\langle y_\beta - y_0, u_\beta - u_0 \rangle| + |\langle y_0 - x, u_\beta - u_0 \rangle| \\ &\quad + |\langle y_\beta - y_0, u_0 \rangle| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

and so  $\lim_{\beta \in \Gamma} \operatorname{Re}\langle y_\beta - x, u_\beta \rangle = \operatorname{Re}\langle y_0 - x, u_0 \rangle$ .

Hence, we have

$$\inf_{u \in f(x)} \operatorname{Re}\langle y_0 - x, u \rangle \leq \operatorname{Re}\langle y_0 - x, u_0 \rangle = \lim_{\beta \in \Gamma} \operatorname{Re}\langle y_\beta - x, u_\beta \rangle \leq \lambda,$$

i.e.,  $y_0 \in A_\lambda$  and  $A_\lambda$  is weakly closed in  $A$ . This shows that for each  $x \in K$ ,  $y \mapsto \varphi(x, y)$  is weakly lower semicontinuous on each weakly compact subset  $A$  of  $K$ .

- (c) There exists a nonempty weakly compact convex subset  $K_0$  of  $K$  and a nonempty weakly compact subset  $D$  of  $K$  such that for each  $y \in K \setminus D$ , there exists  $x \in \operatorname{co}(K_0 \cup \{y\})$  with

$$\operatorname{Re}\langle y - x, u \rangle > 0, \quad \text{for all } u \in f(x).$$

Since  $f(x)$  is strongly compact, therefore  $\varphi(x, y) = \inf_{u \in f(x)} \operatorname{Re}\langle y - x, u \rangle > 0$ .

Now equip  $E$  with the weak topology, and we see that all hypotheses of Lemma 2.1 hold. Hence, it follows that there exists an  $\hat{y} \in K$  such that

$$\varphi(x, \hat{y}) = \inf_{u \in f(x)} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0, \quad \text{for all } x \in K. \quad (3)$$

By Lemma 2.2, we have that

$$\varphi(x, \hat{y}) = \inf_{u \in f(\hat{y})} \operatorname{Re}\langle \hat{y} - x, u \rangle \leq 0, \quad \text{for all } x \in K.$$

Finally, if  $f(\hat{y})$  is convex, define  $g : K \times f(\hat{y}) \rightarrow \mathbb{R}$  by

$$g(x, v) = \operatorname{Re}\langle \hat{y} - x, v \rangle.$$

Then for each  $x \in K$ ,  $v \mapsto g(x, v)$  is weak\* continuous and affine and for each  $v \in f(\hat{y})$ ,  $x \mapsto g(x, v)$  is also affine. By the minimax theorem of Kneser [14] (see also [15]),

$$\min_{v \in f(\hat{y})} \sup_{x \in K} \operatorname{Re}\langle \hat{y} - x, v \rangle = \sup_{x \in K} \min_{v \in f(\hat{y})} \operatorname{Re}\langle \hat{y} - x, v \rangle \leq 0.$$

Since  $f(\hat{y})$  is norm compact, there exists  $\hat{v} \in f(\hat{y})$  such that

$$\operatorname{Re}\langle \hat{y} - x, \hat{v} \rangle \leq 0, \quad \text{for all } x \in K,$$

and so

$$\operatorname{Re}\langle x - \hat{y}, \hat{v} \rangle \geq 0, \quad \text{for all } x \in K.$$

i.e.,  $(\hat{y}, \hat{v})$  is a solution of  $\text{GVIP}(f, K)$ . ■

As a special case of Theorem 2.1, we have the following corollary.

**COROLLARY 2.4.** *Let  $K$  be nonempty compact and convex subset of a Hausdorff topological vector space  $E$ . Let  $f : K \rightarrow 2^{E^*}$  be a quasi-monotone mapping such that  $f$  is upper semicontinuous from the line segments in  $K$  to the weak\* topology of  $E^*$  and each  $f(x)$  is nonempty compact in the strong topology on  $E^*$ . Then there exists a point  $\hat{y} \in K$  such that*

$$\inf_{v \in f(\hat{y})} \operatorname{Re}\langle \hat{y} - x, v \rangle \leq 0, \quad \text{for all } x \in K.$$

*If, in addition,  $f(\hat{y})$  is also convex, then there exists  $\hat{v} \in f(\hat{y})$  such that*

$$\operatorname{Re}\langle x - \hat{y}, \hat{v} \rangle \geq 0, \quad \text{for all } x \in K,$$

*that is,  $(\hat{y}, \hat{v})$  is a solution of  $\text{GVIP}(f, K)$ .*

**REMARK 2.2.** We would like to note that Theorem 2.1 not only shows that Theorem A of Ding [1] is true without the assumption that  $E$  is of secondary category, but it also shows that the conclusion of Theorem A holds in topological vector spaces instead of locally convex topological vector spaces. By following the same ideas used in literature such as [1,2,4,5,7,8,10,15,16] and so on, we can also prove a number of existence theorems of  $\text{GVIP}(f, K)$  and  $\text{GCP}(f, K)$  in which underlying spaces are not of secondary category. We omit the full details here.

Finally, we also wish to discuss the difference between our Theorem 2.1 and Theorem 1 of Cubiotti [17] (which is an extension of [18, Theorem 2.1]) which also concerns the existence of solutions for (GQVI) in finite dimensional spaces. First, we note that Cubiotti's coercive condition (iii) of Theorem 1 in [17] is different from ours in Theorem 2.1. Second, our set-valued mapping  $f$  in Theorem 2.1 is quasi-monotone and upper semicontinuous (thus it may not be lower semicontinuous); however, the corresponding set-valued mapping  $\Gamma$  of Theorem 1 in [17] is lower semicontinuous and its graph is closed (hence, it may not be quasi-monotone nor upper semicontinuous). Furthermore, the underlying space  $E$  of our Theorem 2.1 is a locally convex topological vector space which may be finite or infinite-dimensional, but the underlying space of Theorem 1 (see also [18, Theorem 2.4]) is finite dimensional. Thus, our Theorem 2.1 and Theorem 1 of Cubiotti [17] (see also [18, Theorem 2.4]) are independent each other.

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